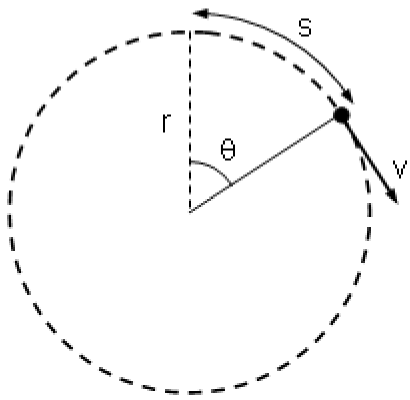
Circular Motion

1 Describing Circular Motion



By definition $\theta = s/r$ (in rad)

Then $d\theta/dt = (ds/dt)/r$

gives $\omega = v/r$

$\theta = s/r$

$\omega = v/r$

1	Arc length (s)	• It is the line segment on the circumference of a circle.
2	Angular Displacement (θ)	• It is the angle swept out by the radius r joining the body to centre of the circle. • $\theta = s/r$ • Unit: *rad*, radian • When arc length is the circumference, $s = 2\pi r$ and $\theta = 2\pi$ rad. Thus 2π rad = 360°.
3	Angular velocity (ω)	• It is the rate of change of angular displacement. • $\omega = d\theta/dt$ or $\Delta\theta/\Delta t$ when $\Delta t \to 0$. • Unit: rad s^{-1} • (Not in syllabus: It is a pseudo-vector perpendicular to the circular plane)
4	Tangential or linear velocity (v)	• It is the *instantaneous velocity* which is the rate of change of linear displacement • $v = ds/dt$ or $\Delta s/\Delta t$ when $\Delta t \to 0$. • A vector with a specific direction. Unit: m s^{-1}
5	Frequency (f)	• It is the number of revolutions per second. • Unit: s^{-1}, hertz or Hz.
6	Period (T)	• It is the time taken for the object to make one complete revolution or cycle. • T = circumference ÷ velocity or $T = 2\pi r/v = 2\pi/\omega$ • From definition of frequency, $T = 1/f$. Combined with $T = 2\pi/\omega$, get $\omega = 2\pi f$

$T = 2\pi r/v$

$T = 1/f$

$\omega = 2\pi f$

2 Centripetal Acceleration and Centripetal Force

Why in *uniform* (i.e. constant speed) circular motion, the object has acceleration?

Although the magnitude of the linear velocity remains constant, its *direction* changes continuously which means the velocity changes continuously. Since $a = \Delta v/\Delta t$, there is acceleration if there is a velocity change. Furthermore, by Newton's Second Law, there must be a net force that is in the direction of the acceleration.

Uniform means constant speed.

Changing *direction* of velocity $\vec{v}$ means velocity is changing, therefore non-zero acceleration.

What is the direction of the acceleration?

Given uniform circular motion, deduce that acceleration and net force are centripetal:

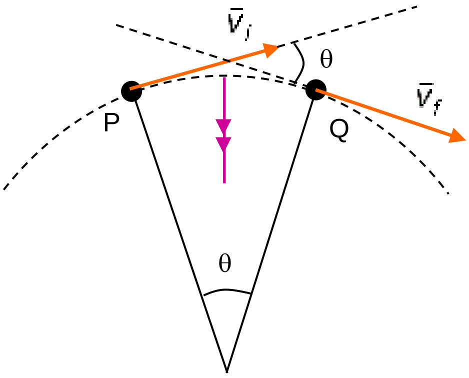


Fig. 1

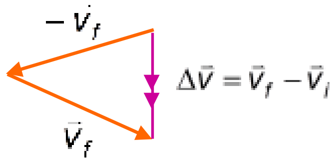


Fig. 2

In Fig. 1, as the particle moves from P to Q (for very small θ and time interval Δt), its tangential velocity changes from $\vec{v}_i$ to $\vec{v}_f$. The vector diagram in Fig. 2 shows that the *change in velocity* (which is a vector) which can be taken to occur at an average position on the arc PQ, is directed towards the centre of the circle. Since acceleration is defined as the rate of *change of velocity*, $\vec{a} = \dfrac{\Delta\vec{v}}{\Delta t}$, acceleration must be in the same direction as the *change of velocity* which is towards the centre of the circle. Hence both the acceleration and net force are *centripetal* (meaning centre seeking).

Given uniform circular motion, $\Delta\vec{v}$ is towards centre of circle based on vector diagram. Then based on definition of $\vec{a}$, it is also towards centre of circle.

Since $\vec{F}_{net} = m\vec{a}$, the net force is also towards centre of circle.

Given F_{net} is always perpendicular to velocity, deduce that motion is circular and speed constant:

- For a net force to be able to modify the magnitude of a velocity, it needs to have a component along the velocity.
- For a net force to *only* change the direction of a velocity, it needs to be *always* perpendicular to the velocity.
- It is useful to visualise the vectors $\vec{v}_f = \vec{v}_i + \Delta\vec{v}$ where $\Delta\vec{v} = \vec{a}\Delta t$. The direction of $\Delta\vec{v}$ must be the same as for $\vec{a}$ and same as for $\vec{F}_{net}$ since $\vec{F}_{net} = m\vec{a}$.

If we start with a moving body and we are able to provide a net force that is constant in magnitude and *always* perpendicular to the velocity, the resulting motion will be uniform and circular. Having the net force *always* perpendicular to the velocity ensures that the speed will not be changed. Keeping the net force constant is to ensure that the radius of the circle does not change.

Given a net force that is constant in magnitude and *always* perpendicular to the velocity, the resulting motion will be uniform and circular.

What does the magnitude of the centripetal acceleration and centripetal force depend on?

The centripetal acceleration is given by (Derivation not expected)

$$a_c = \frac{v^2}{r}$$

Using $v = r\omega$ Using $v = r\omega$

$$a_c = r\,\omega^2 \qquad \text{Using } v = r\omega \qquad a_c = v\,\omega$$

$a_c = v^2/r$
$= r\omega^2$
$= v\omega$

Since F_{net} = ma, centripetal force is

$$F_c = m\,\frac{v^2}{r} \qquad F_c = m\,r\,\omega^2 \qquad F_c = m\,v\,\omega$$

$F_c = mv^2/r$
$= mr\omega^2$
$= mv\omega$

3 More About Centripetal Force

- The equation $F_c = mv^2/r = mr\omega^2 = mv\omega$ is best understood as a way to calculate the centripetal force ***needed*** for uniform circular motion of a mass m moving with speed v in a circle of radius r.

$F_c = mv^2/r$ gives the centripetal force ***needed***.

- If the resultant F_{net} of the forces on the body happens to satisfy the criteria that its direction is at all times perpendicular to the velocity and the magnitude matches the value mv^2/r, then we say that the existing forces are able to ***provide*** the ***needed*** centripetal force to maintain uniform circular motion.

If the F_{net} **provided** becomes greater or less than what is needed, object will veer inwards or outwards.

- If centripetal F_{net} is initially = mv^2/r but later becomes > mv^2/r, the result is that the body will curve inwards towards the centre. If it becomes < mv^2/r instead, the body will curve outwards, e.g. a car encountering a slippery patch while in a circular path will veer outwards due to reduced friction.

- **Centripetal force is not produced by circular motion**. Hence, in any free body diagram, centripetal force should not be drawn in the diagram like any other forces as it is the resultant of all the other forces.

F_c is *not* produced by the circular motion.

- As centripetal force is always perpendicular to the instantaneous displacement, the **work done by centripetal force is always zero**.

Work done by F_c is zero.

- This topic is just an extension of Dynamics. We have seen how a net force can cause an object to slow down or speed up. Now we see that a net force can also cause an object to only change its direction without any change in its speed. In general, a net force can cause either or both a change in speed and direction. Objects in uniform circular motion also serve as a good example that acceleration and velocity need not be in the same direction but acceleration and *velocity change* must be in the same direction.

In general, F_{net} can cause either or both a change in speed and direction.

4 Vertical Circular Motion

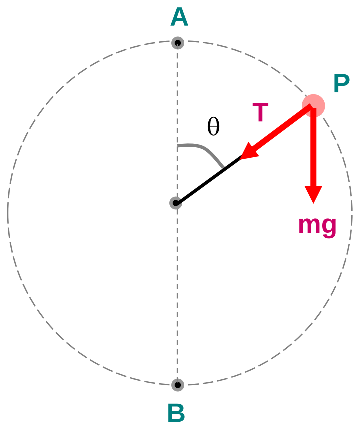
In *most* vertical circular motions the tangential speeds are not uniform so is it still valid to use the equation $F_c = mv^2/r$?

As the object has different speeds at different points in the circle, the equation *must be applied to each moment in time* such that the F_c and v are corresponding values at that same moment. The next moment will involve another set of F_c and v. Furthermore F_c is no longer = F_{net} as there is a tangential force as well.

In non-uniform circular motions, we can use $F_c = mv^2/r$ where the values of F_c and v must correspond for each moment in which the equation is applied to.

Example 1

Consider a weight tied to a string that is fixed at the other end. The weight is given a push such that it goes round in a vertical circle. The weight slows down as it goes higher, speeds up when it comes down:



At P, centripetal force is $F_c = T + mg\cos\theta$ (1)

At special points like A and B, equation (1) reduces to:

At A: $F_c = T_A + mg = mv_A^2 / r$ (2)

At B: $F_c = T_B - mg = mv_B^2 / r$ (3)

Notice that F_c and v are different at different points. As $v_A < v_B$, we can deduce that $T_A < T_B$. In fact, the tension in the string is maximum when the weight is at B and minimum when it is at A.

Example 2

Given that a car does not lose contact at the top of the circular hump, $F_c = mv^2/r$ gives $mg - R = mv^2/r$ (this equation is only describing the moment when car is at the top).