Dynamics

1 Introduction

In Kinematics, we looked at velocities, displacements and trajectories for a body with acceleration at constant direction and magnitude. In Forces, we looked at the different kinds of forces and their characteristics. Now, in Dynamics, we are interested to identify forces at work in a given situation and figure out how the net force determines the acceleration and thus the type of motion or the lack of motion.

2 Newton's Third Law

> When body A exerts a force on body B, body B will exert an equal but opposite force on body A.

Newton's Third Law states that when body A exerts a force on body B, body B will exert an equal but opposite force on body A.

The pair of equal and opposite forces is called an *action-reaction pair* or the *third law pair*. The law essentially says that all forces come in pairs. Hence if Earth pulls on you with a gravitational force, you must be pulling on Earth with the same type of force i.e. gravitational.

What the third law says about the action and reaction pair of forces:

1. they have equal magnitude
2. they have opposite directions
3. they act on different bodies
4. they are of the same type

3 Newton's First Law and Inertia

> Every body continues in its state of rest
> or of uniform motion in a straight line
> until it is caused to change by a *net* force.

Newton's First Law states that every body continues in its state of rest or of uniform motion in a straight line until it is caused to change by a *net* force.

The breakthrough significance of this law may be harder to appreciate in this modern age than when it was first put forth in the 17th century by Newton. Back then, people did not have our present understanding of forces. They attributed the eventual stoppage of everyday moving objects to their *inherent nature* instead of attributing to frictional forces at work.

Today, it is much easier for us to understand that if we remove all forces acting on a body, it will stay at rest if it were originally at rest and it will move with constant speed in a straight line if that was what it was doing before.

However, two aspects of the law are still easily overlooked. The first concerns 'uniform motion in a straight line' which means constant velocity or constant speed and constant direction. If a body moves with constant speed in a circle, it in fact has a changing velocity because the direction of the velocity vector changes continuously and that requires a net force.

A change in motion can mean a change in the magnitude or direction of velocity.

The second aspect is the 'net' force. Occasionally, some people mistake the law to mean if 'no forces' act on a body, then there will be no change to the motion. In fact, 'no forces' is not equivalent to 'no net force'.

Inertia

The First Law also implies that objects have a *tendency* to remain at rest or in constant velocity motion i.e. the tendency to *resist change* in motion. This tendency is called *inertia* and the first law is also called the Law of Inertia. It turns out that a measure of inertia is just mass. Hence a *more massive* object has *greater tendency* to stay at rest or it is harder to get it moving. Similarly, a *more massive* object with a certain velocity has *greater tendency* to maintain that velocity or it is harder - more force required - to cause its velocity to change, be it magnitude or direction.

Inertia is the tendency to resist a change in motion. *Mass* is a measure of inertia.

Fig. 3.1

The driver of a truck carrying logs as shown in Fig. 3.1 must be aware of the logs' inertia because it is a matter of life and death. The driver must secure the logs properly or else during sudden braking, the logs may not stop while the truck has stopped. The driver may then be crushed by the forward moving logs. Such tragic events happen because the *massive* logs have *great* inertia and so they require a large net force to stop. When not properly secured, it means there is insufficient force available to bring the logs to a stop within the short sudden braking time.

The larger the mass, the greater the tendency to resist a change in motion.

You might have experienced being thrown towards the right side when the bus you were in suddenly swerved to the left. This also can be explained using the law of inertia or Newton's First Law. Assume that before the bus turned, you were travelling in a straight line and thus had the tendency or inertia to continue moving forward in a straight line. When the bus turned left, your lower body was in contact with the seat and so the seat dragged your lower body leftwards via friction force. Meanwhile your upper body didn't receive the help of such an external force except the sideways pull from your lower body. As a result, your upper body tended to follow the original straight line motion and so tilted to the right. If the seat were very smooth such that there was very little friction, you would find your whole body sliding towards the right until you hit the right side body of the bus and the normal contact force from the bus then provided a net force to bring about a change in direction of your velocity.

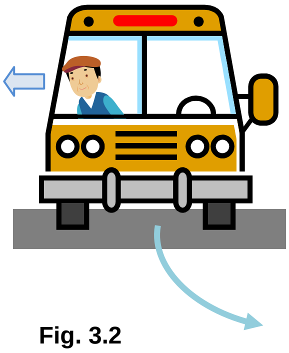


Fig. 3.2

4 System and Free Body Diagram

System

In physics, *clearly identifying* the *system* or the object under study is very important. Identifying a system is like defining a problem. An ill-defined problem or system makes it difficult or impossible to have a clear understanding of the problem or system. Also, an ill-defined problem lacks a clear boundary and so we cannot clearly say whether we have solved it or not because we do not quite know the actual scope of the problem. The principle of clear definition of problem or system is in fact very general and applicable to many non-physics contexts.

Clearly identifying a system or object of interest is critical to analyses and problem solving.

Whoever best describes the problem is the one most likely to solve it.

Dan Roam

If we can really understand the problem, the answer will come out of it, because the answer is not separate from the problem.

Jiddu Krishnamurti

While it is useful to set up a boundary for a problem or system, we must not forget that what is inside may interact or have mutual influence with what is outside. For a system in physics, try and imagine a boundary enclosing the object of interest. Forces and energies originating from outside this enclosure are considered *external*. A system can have quantifiable *properties* like mass, energy, charge, pressure etc. The properties of the system can often be changed through interactions with the external world or they can change due to some internal processes.

Free Body Diagram

In most given situations where we are interested to analyse the connection between the type of motion and the forces at work, we need to be clear which object or part of the object we are interested in i.e. which is the system. To help us to focus only on the system, we make a sketch of the system together with the forces on it. Such a sketch is called a free body diagram.

A free body diagram shows the forces acting on an identified system.

Fig. 4.1

For example, in Fig. 4.1, a boat is travelling with constant velocity. If we are interested in the forces and their moments on the boat, then we would sketch Fig. 4.2, which shows that there are both clockwise and anti-clockwise moments. However, in many cases when we do not need to know the points of application of the forces, we can represent the system by a circle or box and focus solely on showing the directions and relative magnitudes of the forces as shown in Fig. 4.3.

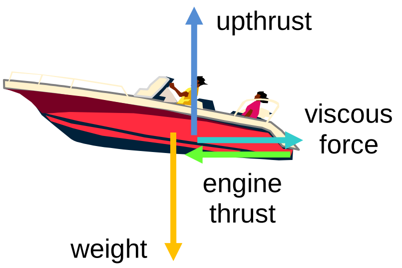


Fig. 4.2

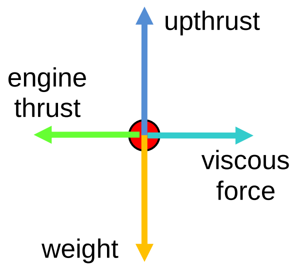


Fig. 4.3

Whether representing a system with a circle or a fairly accurate sketch depends on what needs to be shown.

5 Newton's Second Law

Momentum

Before discussing Newton's Second Law, we need the concept of momentum.

> Linear momentum of a body is defined as the product of its mass and velocity. $\vec{p} = m\vec{v}$

Linear momentum of a body is defined as the product of its mass and velocity. $\vec{p} = m\vec{v}$

It is a rule that multiplying a vector by a scalar gives a vector. Mass m is a scalar while velocity $\vec{v}$ is a vector, hence momentum $\vec{p}$ is a vector.

The reason why $m\vec{v}$ is called *linear* momentum is because there is also *angular* momentum. When not stated, the word momentum is by default referring to linear momentum.

The insensitivity to the vector nature of quantities like velocity and momentum is a major cause of many mistakes made by students. Many difficulties arise from the inability to work out the *change of a vector* quantity. For example, consider a ball with momentum 4 kg m s^{-1} to the right colliding with a wall and rebounding with a momentum of 3 kg m s^{-1} to the left, what is the change in its momentum? If your answer is 'decrease of 1 kg m s^{-1}' then you need to review your concept of vector subtraction. The correct answer is '7 kg m s^{-1} to the left'. Another example below shows how $\Delta\vec{p}$ is obtained.

It is important to be sensitive to the vector nature of momentum and 'change of momentum'.

$\Delta\vec{p} = \vec{p}_f + (-\vec{p}_i)$

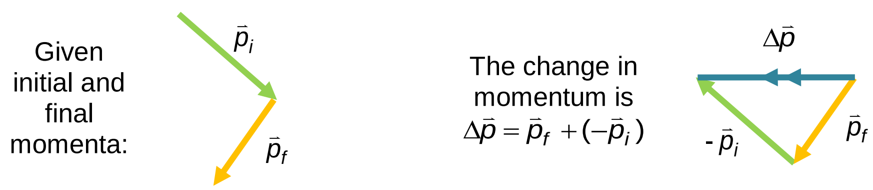


Fig. 5.1

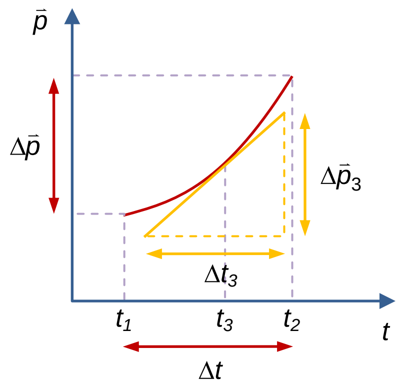
Newton's Second Law

> The law states that the rate of change of momentum of a body is proportional to the net force acting on it and has the same direction as the force. $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$

For problem solving, it is useful to be familiar with other versions of the $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$ equation:

1. $\vec{F}_{net} = m\dfrac{d\vec{v}}{dt} = m\vec{a}$ since $\vec{p} = m\vec{v}$
2. Average net force, $\vec{F}_{net} = \dfrac{\Delta\vec{p}}{\Delta t}$ or $\vec{F}_{net}\Delta t = \Delta\vec{p} = m\Delta\vec{v}$

Average Vs Instantaneous



Average force for $\Delta t = t_2 - t_1$ is = $\dfrac{\Delta\vec{p}}{\Delta t}$

Instantaneous force $\dfrac{d\vec{p}}{dt}$ at t_3 is = $\dfrac{\Delta\vec{p}_3}{\Delta t_3}$

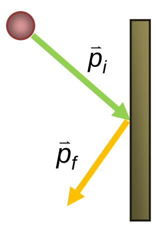
Fig. 5.2

Directions of F_{net}, Δp, Δv, a

It is important to see that 'rate of change of momentum' is a vector. Rate of change of momentum = $\Delta\vec{p}/\Delta t$ is a vector because a vector $\Delta\vec{p}$ divided by a scalar Δt gives a vector. Hence based on the relation $\vec{F}_{net} = d\vec{p}/dt$, the direction of $\vec{F}_{net}$ must be the same as the direction of $\Delta\vec{p}$.

Let's look again at Fig. 5.1. The initial and final momenta could be that of a billiard ball just before and after colliding into a wall at one side of the table. The directions of these momenta are also the directions of the initial and final velocities since $\vec{p} = m\vec{v}$.

Given initial and final momenta:



The change in momentum is $\Delta\vec{p} = \vec{p}_f + (-\vec{p}_i)$

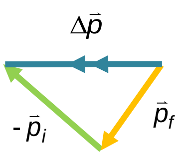


Fig. 5.3

The vector diagram on the right shows the direction of $\Delta\vec{p}$, which must be the direction of the net force acting on the ball. In this case, the net force is a normal contact force by the table on the ball.

Based on the relations $\vec{F}_{net} = d\vec{p}/dt = m(d\vec{v}/dt) = m\vec{a}$, we can in fact say that the directions of $\vec{F}_{net}$, $\Delta\vec{p}$, $\Delta\vec{v}$ and $\vec{a}$ are *always* the same.

Newton's Second Law states that the rate of change of momentum of a body is proportional to the net force acting on it and has the same direction as the force.

$\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$

Also,

$\vec{F}_{net} = m\dfrac{d\vec{v}}{dt} = m\vec{a}$

ave $\vec{F}_{net} = \dfrac{\Delta\vec{p}}{\Delta t}$

Second law is used to define force:
Force on a body is defined as the rate of change of momentum of the body in the direction of the force.

Based on the relations

$\vec{F}_{net} = d\vec{p}/dt$
$= m(d\vec{v}/dt)$
$= m\vec{a}$

the directions of $\vec{F}_{net}$, $\Delta\vec{p}$, $\Delta\vec{v}$ and $\vec{a}$ are *always* the same.

Impulse-momentum relation $F_{net}\Delta t = \Delta p$

When the second law is written as $\vec{F}_{net}\Delta t = \Delta\vec{p} = m\Delta\vec{v}$, it is also known as the *impulse-momentum relation*. When you apply a *net* force $\vec{F}_{net}$ for duration of time Δt on an object, it is said that you have applied an impulse of $\vec{F}_{net}\Delta t$ on the object. As a result of that impulse, the momentum of the object would have changed by $\Delta\vec{p}$ or $m\Delta\vec{v}$. Put in another way, when a net force acts on an object, its velocity must change in either magnitude or direction or both. This is what the first law is saying but the second law says the same and more by quantifying and relating the quantities $\vec{F}_{net}$, Δt, m and $\Delta\vec{v}$.

$\vec{F}_{net}\Delta t$ is called impulse (vector).

$\vec{F}_{net}\Delta t = \Delta\vec{p} = m\Delta\vec{v}$ is the impulse-momentum relation.

Note that $\vec{F}_{net}$ must either be an average value or a constant value during Δt. If $\vec{F}_{net}$ were not constant as shown in the following graph, then the impulse is equal to the area under the graph.

$\vec{F}_{net}\Delta t = \Delta\vec{p}$ is true only for constant net force or average net force. If $\vec{F}_{net}$ is not constant, then impulse and momentum change are both given by the area under the $\vec{F}_{net} - t$ - t graph.

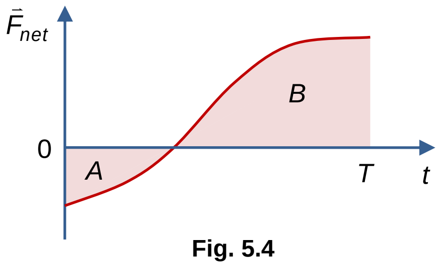


Fig. 5.4

Total impulse during T = area B - area A

(Average $\vec{F}_{net}$)T = area B - area A

Average $\vec{F}_{net}$ = net area / T

Imagine that 5 bullets are fired from a machine gun in quick succession and each experiences a horizontal stopping force ($\vec{F}_{net}$) exerted by a wall and the forces are plotted on a time axis:

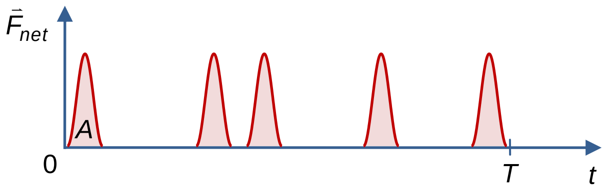


Fig. 5.5

The area A for each bullet is the impulse exerted by the wall on the bullet to stop it. By Newton's Third Law, the impulse (magnitude only) exerted by each bullet on the wall is also A. The total impulse is thus 5A over the time duration T. The average force exerted on the wall by the bullets over time T is thus = 5A/T or the total momentum change of the 5 bullets divided by the time T.

6 Using Second Law

The skilful use of second law to solve problems requires the following competencies:

I. mastery of general problem solving approach
II. clear identification of system
III. familiarity with the types and characteristics of forces
IV. correct understanding and handling of vectors

Example 1

A box of weight W is resting on a slope of inclination θ and we want to find the friction force on the box.

What would be the first thing we do?

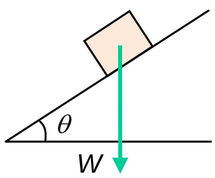


Fig. 6.1

If we can really understand the problem, the answer will come out of it, because the answer is not separate from the problem.

Jiddu Krishnamurti

The first thing to do is to *understand the problem*. In this example, the word *resting* in ‘weight W resting on a slope’ is a word that holds key meaning. The word means that the box has zero velocity *and* it continues to be at rest at subsequent times i.e. it has no change in velocity and so it has zero acceleration. Zero acceleration in turn implies that $\vec{F}_{net} = m\vec{a} = 0$. Now $\vec{F}_{net} = 0$ is the equation we can use to find friction force!

Part of the approach towards any problem is to always understand the problem.

That understanding will naturally allow you to draw on relevant knowledge for problem solving.

Key words like ‘resting’ could have been easily overlooked without enough practice or exposure to develop the sensitivity. That sensitivity in turn must rest on solid conceptual understanding of the word and its associated ideas.

In general, a problem of finding an unknown almost always involves using an equation or equations. The choice of each equation will depend on the given situation since each equation is only applicable to a certain context. With practice and understanding, you will be able to associate the situation in a problem to relevant equations.

In this current example, it is not difficult to see that the system should be the box after we come to the conclusion that it must satisfy the equation $\vec{F}_{net} = 0$. Since the situation is simple - only a box and a slope - a free body diagram would not be necessary. Instead, the forces on the box can be directly drawn on the diagram as shown in Fig. 6.2.

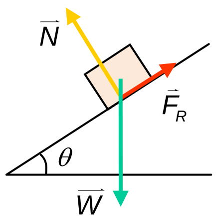


Fig. 6.2

In physics problems, being clear about the system to analyse is critical. In the usage of the second law, the lack of a clear object of interest would impact the correct identification of relevant forces contributing to $\vec{F}_{net}$.

The equation $\vec{F}_{net} = 0$ in this context means $\vec{N} + \vec{W} + \vec{F}_R = 0$ which does not mean that the addition of the magnitudes of the forces is equal to 0. It means that if we were to join the vectors head to tail, they will form a closed triangle without any resultant vector as shown in Fig. 6.3.

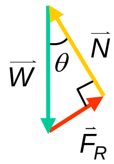


Fig. 6.3

Based on the triangle in Fig. 6.3, we can then deduce that $\left|\vec{F}_R\right| = \left|\vec{W}\right| \sin\theta$ or $F_R = W \sin\theta$.

If the problem had involved the same box sliding down the slope at constant velocity, then the answer $F_R = W \sin\theta$ will still be valid. This is because ‘constant velocity’ implies $\vec{F}_{net} = 0$ also and all the steps to arrive at the answer will be the same. For comparison, this problem has also been solved by two methods under the chapter on Forces in section 12 on Equilibrium.

Example 2

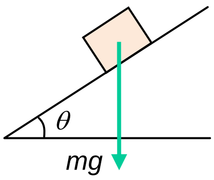


Fig. 6.4

A box of mass *m* has acceleration *a* down a slope of inclination *θ*. Find the friction force on the box.

Similar to Example 1 but $\vec{F}_{net} \neq 0$

Given a is downslope $\Rightarrow$ $\vec{F}_{net}$ downslope too.

Vector diagram now looks as shown in Fig. 6.5.

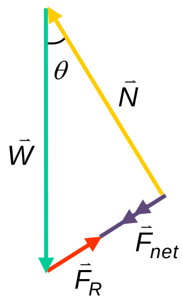


Fig. 6.5

Focus on *magnitudes* in Fig. 6.6.

Since no acceleration perpendicular to the slope, the magnitudes $N = W\cos\theta$.

On the other hand, second law requires $F_{net} = ma$ downslope,
so $W\sin\theta - F_R = ma$,
$\therefore F_R = \mathbf{W\sin\theta - ma}$

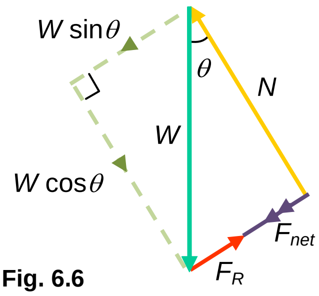


Fig. 6.6

Example 3

A boy stands on a weighing scale inside an elevator that is *moving upwards* and *slowing down* at a rate of 2 m s^{-2}. If the boy's mass is 50 kg, what is the reading on the scale?

Firstly, understanding the problem:

1. "slowing down" $\Rightarrow$ decelerating $\Rightarrow$ $\vec{F}_{net}$ and $\vec{a}$ both opposite in direction to that of $\vec{v}$ (see Kinematics pg. 3). $\therefore$ $\vec{F}_{net}$'s direction is downward & $\therefore W_B > N_B$.
2. Scale works because of force pressing against it i.e. it measures the normal contact force which is equal and opposite to $\vec{N}$ and translates it into reading of mass in kg.

The system is identified to be the boy because a) his weight is given and b) the scale measures his normal contact force which we want to find.

The force diagram is as shown in Fig. 6.7.

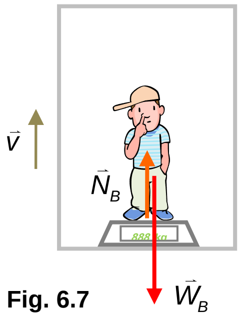


Fig. 6.7

Using the equation $\vec{F}_{net} = m\vec{a}$

$$\vec{W}_B + \vec{N}_B = m\vec{a}$$

Taking downward as positive and replacing each vector in the above equation with a direction (+- sign) and a magnitude,

$$(+mg) + (-N) = m(+a)$$
$$N = mg - ma$$
$$= 50(9.81 - 2)$$
$$= 390.5 \text{ N}$$

This force corresponds to a mass of 390.5/9.81 = 39.8 kg
$\therefore$ the reading on the scale is **39.8 kg**

If this problem had the elevator moving upwards and speeding up, then $N > W$. The weight does not change as m and g are constant. N is the one that can change its magnitude and thus the scale reading will be different depending on whether the acceleration vector is upward or downward.

7 Second Law & Momentum Conservation

The total momentum of a system remains constant
provided no external net force acts on the system.

The above is called the Principle of Conservation of Momentum.

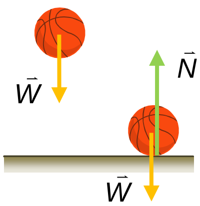
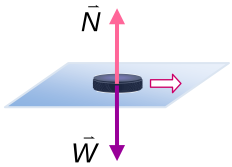
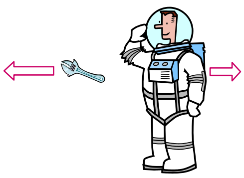
From the second law $\vec{F}_{net} = \frac{d\vec{p}}{dt}$, if the net force on a system is zero, it follows that the rate of change of momentum must be zero, which just means that the momentum of the system does not change with time.

The Principle of Conservation of Momentum (PCOM) states that the total momentum of a system remains constant provided no external net force acts on the system.

The principle applies to a system of a single body or a collection of bodies. For a collection of bodies, the total momentum must be found by adding the individual momenta as vectors.

The principle is one of the most universal in physics and there has not been any situation in which the principle is not applicable. Below are some examples of how the principle is applied to arrive at the conclusion of whether momentum of the system is constant or not.

PCOM is universal i.e. valid for any system at all times.

Situation	System	System's momentum
A ball bouncing on the floor.	Ball 	Ball's **momentum is not constant** because there is mostly **a net force acting on it**. Net force is weight when ball is in the air. Net force is the resultant of the normal contact force and weight when on floor.
An ice hockey puck sliding horizontally across the ice rink at constant velocity.	Puck 	Assume no friction. **No net force** both vertically and horizontally so the puck moves with **constant momentum**.
An astronaut throws a spanner.	Astronaut and spanner 	Assume astronaut-spanner system **does not experience a net force** such as gravitational pull from a nearby planet. Though the astronaut and spanner individually undergoes a change in momentum during the throw, the **total momentum of the system is constant at all times**.

In reality, it is often very difficult to have zero net force on a system. Hence, we often simplify our analyses by ignoring forces that are much smaller compared to the others e.g. the friction on the ice hockey puck. Recall that $F_{net}\Delta t = \Delta p$, if F_{net} and Δt are both small, the effect i.e. momentum change, will be small.

8 One Dimensional Two-body Interactions

Momentum Conservation

In the previous section we saw that a system with zero net force from its surroundings must have constant or conserved momentum. For a 2-body system with zero net force, the individual bodies can interact by exerting forces on each other and hence change their momenta but the system as a whole must have constant momentum. We shall derive the equations describing reflecting this outcome.

For a start, we will look at situations involving an isolated system of 2 bodies free from any influence or forces from the surrounding.

The first example involves 2 bodies moving along the same straight line, approaching each other, collide and then either separate or move off together:

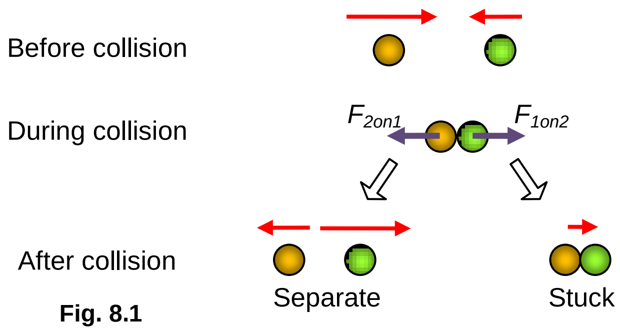


Fig. 8.1

A second example involves 2 bodies initially at rest together but later, due to events such as the release of a compressed spring or an explosion, they separate:

Bullet ejected from toy gun using a compressed spring

Gas ejected from rocket at high speed when fuel burns and expands

Fig. 8.2

In both examples, during the interaction or collision, the 2 bodies exert *equal but opposite forces* on each other (Newton's third law) for the *same duration*.

The forces they exert on each other typically vary as in Fig. 8.3. However, we will simplify things by using a *constant* average force <*F*> in the following analysis.

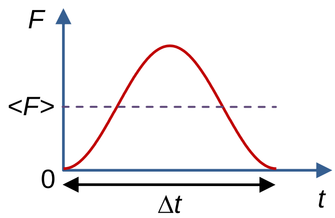


Fig. 8.3

Due to our assumption that there is negligible or no external forces, the only forces are the pair of equal(magnitude) and opposite(directions as indicated by +- signs) forces between the two bodies:

By Newton's 3rd law, $<\vec{F}>_{2on1} = -<\vec{F}>_{1on2}$

Same duration of action so $<\vec{F}>_{2on1}\,\Delta t = -<\vec{F}>_{1on2}\,\Delta t$

Using Newton's 2nd law, $\Rightarrow \boxed{\Delta\vec{p}_1 = -\Delta\vec{p}_2}$

The conclusion is that in the type of interactions considered, the momentum changes of the 2 bodies are equal in magnitude but opposite in direction. Furthermore,

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2$$
$$\Rightarrow \quad \vec{p}_{1f} - \vec{p}_{1i} = -(\vec{p}_{2f} - \vec{p}_{2i})$$
$$\Rightarrow \quad \boxed{\vec{p}_{1f} + \vec{p}_{2f} = \vec{p}_{1i} + \vec{p}_{2i}}$$

The last line says that the total momentum of the 2 bodies finally (at *any* later time) is the same as the total momentum of the 2 bodies initially (at an earlier time). In other words, the momentum of the system is conserved as expected of a system with zero external net force.

In our derivation, we considered situations with *no external forces* but what if there are external forces? If external forces are considered, as long as they add up to zero, the momentum of the 2 body system will still be conserved.

In a system of 2 colliding bodies or of repulsive separation of a body into 2, if there are no external forces on the system, the equations hold:

$\vec{F}_{2on1} = -\vec{F}_{1on2}$

$\vec{F}_{2on1}\Delta t = -\vec{F}_{1on2}\Delta t$

$\Rightarrow \Delta\vec{p}_1 = -\Delta\vec{p}_2$

$\Rightarrow \vec{p}_{1f} - \vec{p}_{1i} = -(\vec{p}_{2f} - \vec{p}_{2i})$

$\Rightarrow \vec{p}_{1f} + \vec{p}_{2f} = \vec{p}_{1i} + \vec{p}_{2i}$

The last line being the *conservation of momentum equation* (COM).

Elastic and Inelastic Collisions

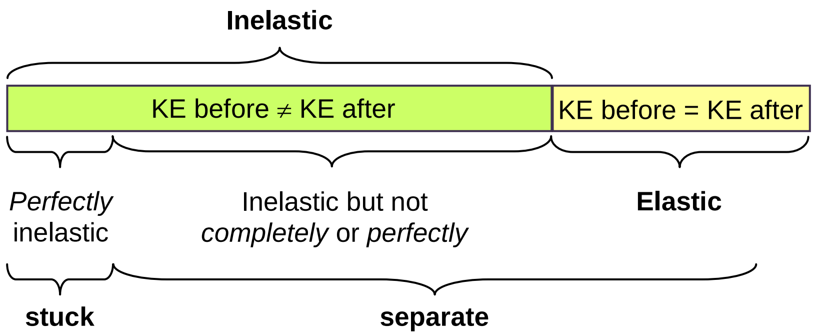
Collisions can be classified as elastic or inelastic:

> An elastic collision is one where the total kinetic energy of the bodies *before* collision is equal to the total kinetic energy *after* collision.
>
> $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

> An inelastic collision is one where the total kinetic energy of the bodies *before* collision is not equal to the total kinetic energy *after* collision.
>
> $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \neq \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

An elastic object is one that is able to return to its original shape after deformation. This elasticity accounts for the conversion of kinetic energy to elastic potential energy during collision and the subsequent reverse conversion. This elasticity or the tendency for the colliding objects to push back after deformation also explains why elastic objects separate after collision.

The following diagram shows the relationship between the type of collision and whether the colliding bodies separate after collision:



Elasticity comes in varying degrees. Hence an elastic collision will have equal kinetic energies before and after collision. An inelastic collision in which 90% of the original total kinetic energy is retained after collision is closer to an elastic collision than to a perfect inelastic collision. In reality, there are very few elastic collisions as most collisions involve the loss of some of the initial total KE.

Sometimes it is said that 'KE is conserved in elastic collisions' which is inaccurate. It is best not to say that because the 'conservation' of KE is only limited to *before* and *after* collision. During collision, the total KE of the system is definitely not the same as before collision because some KE has been converted to elastic PE or other forms of energy.

It is useful to think of non-collision type of interactions, such as the rifle-bullet and the rocket-exhaust examples, as inelastic because they result in final KE of the system being greater than the initial KE.

Relative Speeds - RSA=RSS

Just for emphasis, Section 8 deals with systems isolated from external forces and hence F_{net} = 0 and so the *momentum of the system is conserved at all times*. In addition, we only consider a simple *2-body interaction* involving *1-dimensional motions*. It may seem useless to study such a narrowly defined scenario but the idea is to learn some general principles that can be extended to 2D/3D motions and applicable approximately to real life situations.

In elastic collision: total kinetic energy *before* = total kinetic energy *after* collision.

In inelastic collision: total kinetic energy *before* ≠ total kinetic energy *after* collision.

Colliding bodies separate ⇒ elastic or inelastic.

Bodies stuck together ⇔ perfect inelastic.

Conservation of KE in elastic collisions doesn't apply *during* collisions.

In our narrowly defined context, the conservation of momentum (COM) equation holds:

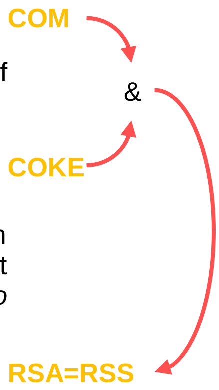
$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

In addition, if the interaction is elastic, the conservation of KE (COKE) equation applies:

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

It can be shown that a third equation can be derived from the above 2 equations. This third equation states that *relative speed of approach* of the 2 bodies *is equal to* their *relative speed of separation*.

RSA = RSS



RSA=RSS equation is derived from the COM equation and the COKE equation.

The version of RSA=RSS in terms of relative *velocities* is $\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$. Using this alternative version requires caution in remembering the order of the subscripts and in handling vectors properly(using +- signs).

$\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$ is the vector version of RSA=RSS

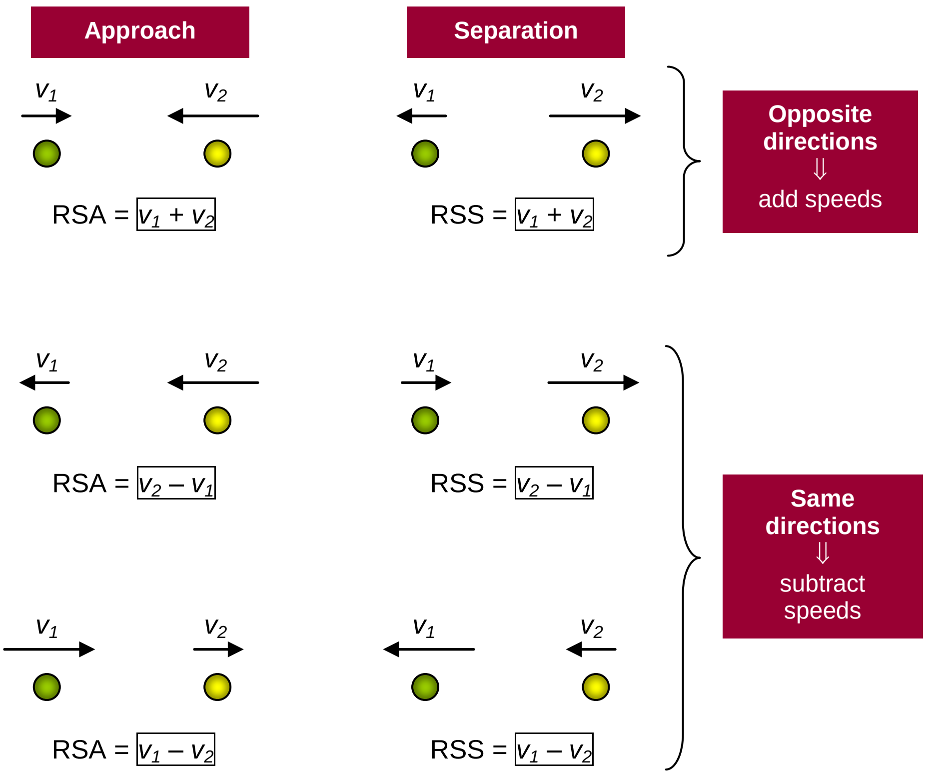
To sum up, for elastic interactions, we can use any two of the three equations above to solve problems. However, the recommended way is to use COM and RSA=RSS, avoiding COKE because a quadratic equation is generally more troublesome to handle. For inelastic interactions, only the COM equation can be used.

Finding Relative Speeds

The diagram below shows all the possible ways of approach and separation as seen by someone outside the 2-body system. To work out the relative speeds, imagine shrinking and putting yourself inside the system by sitting on one of the balls to measure the speed at which the other ball is approaching or separating from you.

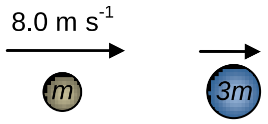
Note that v_1 and v_2 are speeds (positive numbers). Relative speeds must also be positive so you always subtract a smaller speed from a larger speed.

Key to finding relative speeds is to imagine the measurement or perception of speed of one body from the other body.

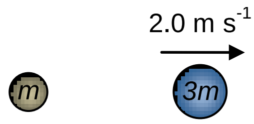


Example 4

A ball of mass m approaches a second ball of mass $3m$ moving to the right as shown.



After the head-on elastic collision, the second ball has velocity of 2.0 m s^{-1} to the right as shown.

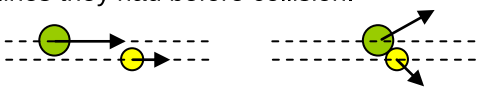


Find the velocity of the first ball after the collision.

Solution:

Step 1 – Understanding the problem

- Check that the system is not experiencing a net force. In this case, the given diagram shows a 2-body system in isolation from any other objects, so there are no external forces at all and so the net force is taken to be zero. Hence, the COM equation applies here.
- Next check that the motions are 1D. Here 'head-on' collision means that the 2 balls approach each other along a line joining their centres of mass. This will ensure that they continue to stay along the line after the collision. In contrast, a *glancing* collision as shown below will result in the balls moving off the lines they had before collision.



  When motions are 1D, the velocities are all either pointing one direction or the opposite direction. This allows us to use +- signs to indicate the directions of vectors in the COM and RSA=RSS equations later.
- Identify whether the collision is elastic or inelastic. Elastic collisions allow us to use the COKE and RSA=RSS equations.

In *head-on* collision the bodies' motions stay along a single straight line. In *glancing* collision, the bodies' lines of motion after collision will not coincide with their original ones.

Generally for elastic collisions, COKE equation is not favoured because of greater difficulty in dealing with squared terms.

Chosen equations:

(1) $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$ COM

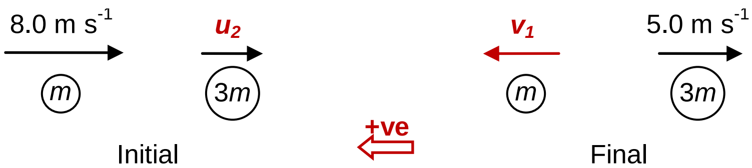
OR $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$

(2) $\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$ RSA=RSS (COKE not favoured)

Step 2 – Planning the solution

- Choose a positive direction for vectors.
- Assume a velocity direction if any is unknown.
- Introduce needed variables - let u_2 be ball 2's initial *speed* and v_1 to be ball 1's final *speed*. It is very important to be aware that we are taking the variables u_2 and v_1 to be speeds(+ve numbers) only while their directions are separately indicated by the arrows.
- A sketch is recommended to help keep track of the choices made:

Assume positive direction is 'toward the left'.
Assume the first ball's final velocity is *to the left*:



Step 3 – carrying out the plan

- Each vector must be substituted with a magnitude accompanied by + or - sign for direction. e.g. $\vec{u}_1 = -8$ below.

(1) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ COM

(1.1) $m(-8) + 3m(-u_2) = m(+v_1) + 3m(-5)$

(1.2) $3u_2 + v_1 = 7$

(2) $\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$ RSA=RSS

(2.1) $(-8) - (-u_2) = (-5) - (+v_1)$

(2.2) $u_2 + v_1 = 3$

(1.2) - (2.2): $2u_2 = 4$

$u_2 = 2.0$ m s^{-1}

Substitute $u_2 = 2.0$ m s^{-1} in (2.2) gives $v_1 = 1.0$ m s^{-1}.

Hence velocity of the first ball after collision is **1.0 m s^{-1} to the left**.

Alternative RSA=RSS

Equation (2) above is strictly speaking not RSA=RSS because the subtraction of two velocities is a relative velocity instead of relative speed. The method to find RSA and RSS is on pg 11.

If we use the relative speeds then the presentation will look like:

(1) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$

(1.1) $m(-8) + 3m(-u_2) = m(+v_1) + 3m(-5)$

(1.2) $3u_2 + v_1 = 7$

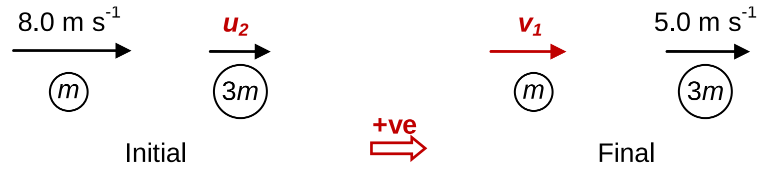
(2) Relative speed of approach = relative speed of separation

(2.1) $8 - u_2 = 5 - v_1$ (see pg 11 on finding relative speeds)

(2.2) $u_2 + v_1 = 3$

Alternative Working 1

Let's consider what happens if the following choices were made:



(1) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$

(1.1) $m(+8) + 3m(+u_2) = m(+v_1) + 3m(+5)$

(1.2) $3u_2 - v_1 = 7$

(2) $\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$

(2.1) $(+8) - (+u_2) = (+5) - (+v_1)$

(2.2) $v_1 - u_2 = -3$

(1.2) + (2.2): $2u_2 = 4$

$u_2 = 2.0$ m s^{-1}

Substitute $u_2 = 2.0$ m s^{-1} into (2.2) gives $v_1 = -1.0$ m s^{-1}. v_1 is a speed and is supposed to be positive. Is there something wrong? Yes, the direction assumed for v_1 is wrong i.e. the **negative sign is an indication that the assumed direction is wrong**.

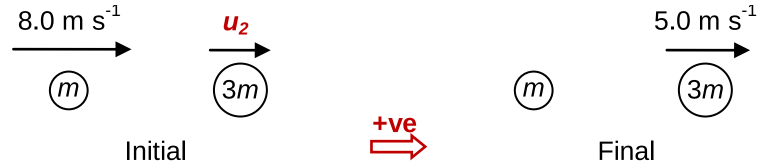
Hence velocity of the first ball after collision is **1.0 m s^{-1} to the left**.

Familiarity with usage of +- signs in relation to vector quantities is crucial in problem solving.

The use of +- signs must be consistent with the directions assumed.

Alternative Working 2

Here, we don't assume a direction for first ball after collision. Thus we leave $\vec{v}_1$ as a variable with unknown direction and magnitude in the equations.



(1) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ COM

(1.1) $m\,(+8) + 3m\,(+u_2) = m\,\vec{v}_1 + 3m\,(+5)$

(1.2) $3u_2 - \vec{v}_1 = 7$

(2) $\vec{u}_1 - \vec{u}_2 = \vec{v}_2 - \vec{v}_1$ RSA=RSS

(2.1) $(+8) - (+u_2) = (+5) - \vec{v}_1$

(2.2) $\vec{v}_1 - u_2 = -3$

(1.2) + (2.2): $2u_2 = 4$ ----- (3)

$u_2 = 2.0$ m s^{-1}

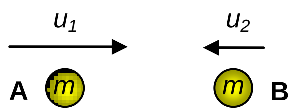
Substitute u_2 = 2.0 m s^{-1} in (2.2) gives $\vec{v}_1$ = -1.0 m s^{-1}. Now, **the negative sign tells us that $\vec{v}_1$ points in the negative direction which is to the left.**

The various alternative workings show the different thinking behind the usage of symbols. In math and physics, symbolic presentation is pervasive. Failure to understand the subtle difference between writing things one way and another is often the cause of much confusion.

Most authors use v for speed and $\vec{v}$ for velocity. However, some people use v to mean either and it is up to readers to figure out what each symbol represents.

Example 5

Two spheres **A** and **B** of mass m move towards each other with speeds as shown.



(i) If u_1 = 4.0 m s^{-1} and u_2 = 2.0 m s^{-1}, find the velocity of sphere **A** when sphere **B** is momentarily at rest.

(ii) Show that in general, if the spheres have equal masses and the collision is head-on elastic, then after the collision each sphere will have a velocity which is the other's initial velocity.

Solution:

Taking +ve to be rightwards.

(i) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ COM

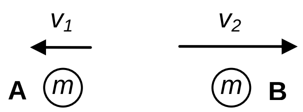
$m\,(4) + m\,(-2) = m\,v_A + 0$

$\mathbf{v_A}$ **= 2.0 m s^{-1}** (Note that v_1 and v_2 can be velocities during collision)

(ii) $m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ COM

$m\,u_1 + m\,(-u_2) = m\,(-v_1) + m\,(v_2)$

$u_1 - u_2 = -v_1 + v_2$ ----- (1)



Assumed outcome

RSA = RSS

$u_1 + u_2 = v_1 + v_2$ ----- (2)

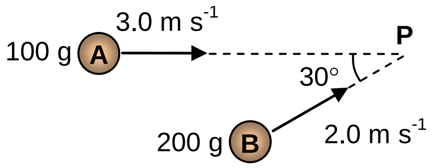
(1) + (2) gives $2u_1 = 2v_2$ or $\mathbf{v_2 = u_1}$

(1) - (2) gives $2u_2 = 2v_1$ or $\mathbf{v_1 = u_2}$ } **Shown**

Outcome of head-on collisions between identical spheres is sometimes assumed to be well-known by students.

Example 6

A and **B** are two discs gliding freely on an air-table. **A** has mass 100 g while **B** has mass 200 g. Their velocities are as shown.

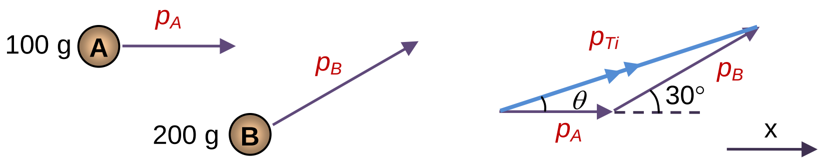


The discs' rims are wrapped with velcro such that when they collide at point **P**, they stick and move off together. Find their common velocity after collision.

Solution:

Discs stick together after collision ⇒ perfectly inelastic collision.
Net force = 0 ⇒ total momentum p_{Ti} before = total momentum p_{Tf} after.
A vector diagram is needed.

For inelastic collisions, COKE and RSA=RSS cannot be used.



$p_{Ti}^2 = p_A^2 + p_B^2 - 2p_Ap_B \cos150°$ Using cosine rule
$p_{Ti}^2 = (0.3)^2 + (0.4)^2 - 2(0.3)(0.4)\cos150°$
$p_{Ti} = 0.677$ kg m s^{-1}

$p_{Tf} = p_{Ti}$ (because COM)
$\therefore (m_A + m_B)v_f = 0.677$ v_f is common speed after collision
$v_f = 2.25$ m s^{-1}

$$\frac{p_{Ti}}{\sin150°} = \frac{p_B}{\sin\theta}$$ Using sine rule
$\theta = 17°$

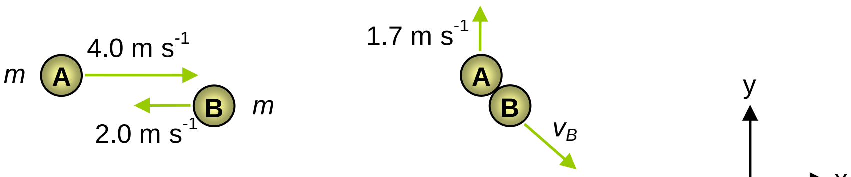
∴ common velocity of discs after collision is **2.3 m s^{-1}** at an **angle of 17° from the x-direction**.

Example 7

The velocities of two identical spheres **A** and **B** before collision are along the x-axis as shown. After collision, **A**'s velocity is in the y-direction.

(i) Find the speed v_B.

(ii) Determine if the collision is elastic.



Before After

Solution:

(i) Net force = 0 ⇒ momentum must be conserved in both x & y directions.
Along x: Total momentum before = 2m, ∴ x component of $\vec{v}_B$ = 2 m s^{-1}
Along y: Total momentum before = 0, ∴ y component of $\vec{v}_B$ = 1.7 m s^{-1}
$v_B = \sqrt{2^2 + 1.7^2}$ = **2.6 m s^{-1}**

(ii) Initial total KE = ½m(4^2 + 2^2) = 10m. Final total KE = ½ m(1.7^2 + 2.6^2) = 4.83 m. Since the final KE < initial KE, collision is **inelastic**.

In a *glancing* collision, motions will be 2D. The approach is to select two perpendicular directions (x, y) and apply the same analyses *separately* to each direction just like in projectile motion.